

COMPUTATIONAL STABILITY ANALYSIS OF LOTKA-VOLTERRA SYSTEMS

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This paper concerns the computational stability analysis of locally stable Lotka-Volterra (LV) systems by searching for appropriate Lyapunov functions in a general quadratic form composed of higher order monomial terms. The Lyapunov conditions are ensured through the solution of linear matrix inequalities. The stability region is estimated by determining the level set of the Lyapunov function within a suitable convex domain. The paper includes interesting computational results and discussion on the stability regions of higher (3,4) dimensional LV models as well as on the monomial selection for constructing the Lyapunov functions. Finally, the stability region is estimated of an uncertain 2D LV system with an uncertain interior locally stable equilibrium point.

Keywords: nonlinear systems, Lotka-Volterra models, stability analysis, linear matrix inequalities, Lyapunov function

1. Introduction

Approximating the domain of attraction (DOA) is often a fundamental task in the analysis and control of nonlinear systems. The stability properties of dynamical systems are most often studied using Lyapunov functions. Therefore, extensive literature exists on the computational construction of Lyapunov functions [1].

Due to their advantageous properties and the availability of efficient numerical solvers, the use of linear matrix inequalities (LMI) and semi-definite programming (SDP) techniques has become popular in the field of system and control theory. Important results were announced [2,3] in the context of linear uncertain systems, their stability analysis and control synthesis.

Recently, an optimization-based method for DOA estimation was published [4], where the authors use Finsler's lemma and affine parameter-dependent LMIs to compute rational Lyapunov functions for a wide class of locally asymptotically stable nonlinear systems. Based on these results an improved method was published [5,6], where the transformation of the model to the form required for optimization is done automatically using the linear fraction transformation (LFT) and further automatic model simplification steps, which results in the dimension reduction of the optimization task. As the dimensions of the problem are

reduced, the method is capable of handling more complex models. For example, the DOA is successfully estimated for a 3-dimensional rational uncertain system [6] (bioreactor model with an applied proportional and integral substrate feedback law).

The dynamical descriptive power of Lotka-Volterra systems is so extensive that LV models "have the status of canonical format" within the class of smooth nonlinear dynamical systems [7]. Besides modelling biological/ecological environments, they are widely used in other scientific fields like neural networks [8] or in economics, where the Goodwin-Lotka-Volterra models are applied for modelling the predator-prey mechanism of the technological substitution [9,10]. In order to model the correlation between the employment rate and the share of wages of the working population, the authors of Ref. [11] used a stochastic extension of the Goodwin model.

Some important results that make LV models even more attractive are the existing techniques used to represent a general nonlinear system as a multidimensional LV model [12]. The analysis of stability and behaviour of LV systems is extensive in the literature [13]. Plank has shown [14] that N -dimensional LV systems are Hamiltonian if they fulfil certain algebraic properties (Theorem 3.1 in Ref. [14]). He demonstrated that, when using an appropriate Poisson structure, one can obtain the Hamiltonian function of the system, which is a key object in determining the system's DOA. Furthermore, candidates of Hamiltonian function were defined in (Section IV in Ref. [14]). It is important to note that LV systems are

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kinetic in the sense that they can be formally described as chemical reaction networks with mass action kinetics.

In this paper, the local stability properties of LV models are analysed by applying the underlying method presented [6]. The stability region is estimated on a few locally stable 2-, 3-, and 4-dimensional benchmark LV systems. The DOA of an uncertain 2D system is also estimated by two concentric regions.

The main motivation of our work was to evaluate the applicability of the approach [4] on a general polynomial system class consisting of low-degree monomials, and to study the limits of the method as the number of dimensions of the state space increase.

2. Background

In this section, the basic notions and results on which our computational method is based are presented.

2.1. System Class, Lyapunov Functions, and the Domain of Attraction.

Our computational method can handle general nonlinear systems of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), \delta(t)) \\ x(t) &\in \mathbb{R}^N, x_0 \in \mathcal{X}, \delta(t) \in \mathcal{D}, \dot{\delta}(t) \in \tilde{\mathcal{D}} \end{aligned} \quad (1)$$

where $\mathcal{X} \subset \mathbb{R}^N$ and $\mathcal{D}, \tilde{\mathcal{D}} \subset \mathbb{R}^D$ are given polytopes, x is the state vector function with its initial condition $x_0 = x(0)$, and δ is a smooth, bounded vector function of uncertain parameters with a bounded time derivative. The applied method is presented in detail elsewhere [15]. From now on, the time arguments of x and δ will be suppressed as is commonly done in the literature. It is assumed that function $f: \mathbb{R}^N \times \mathcal{D} \mapsto \mathbb{R}^N$ of Eq.(1) is a well-defined smooth *rational mapping*, with the property $f(0, \delta) = 0$ for all $\delta \in \mathcal{D}$. Additionally, it is assumed that $x^* = 0 \in \mathbb{R}^N$ is a locally asymptotically stable equilibrium point of Eq.(1) for all $\delta \in \mathcal{D}$. The set of all initial conditions, from which the solutions converge to x^* , is called the *domain of attraction* (DOA). Furthermore, it is assumed that function $f(x, \delta)$ can be written in a so-called quasi-LPV form $f(x, \delta) =$ function $A(x, \delta) x$

The aim is to identify an appropriate rational Lyapunov function $V(x, \delta)$, which satisfies the following conditions:

$$\begin{aligned} v_1(x) &\leq V(x, \delta) \leq u_x(x), \forall (x, \delta) \in \mathcal{X} \times \mathcal{D} \\ \dot{V}(x, \delta, \dot{\delta}) &\leq -v_d(x) \forall (x, \delta, \dot{\delta}) \in \mathcal{X} \times \mathcal{D} \times \tilde{\mathcal{D}} \end{aligned} \quad (2)$$

where v_1, u_x, v_d are continuous positive functions on \mathcal{X} . Due to conditions of Eq.(2) any closed level set of the Lyapunov function contained entirely in \mathcal{X} bounds an invariant region of the state space. Our objective is to find a Lyapunov function, which fulfils the conditions of Eq.(2) and to determine its maximal invariant level set completely inside \mathcal{X} .

2.2. Dynamical System Representation

A Lyapunov function can be computed [4] in the form shown in Eq.(3).

$$V(x, \delta) = \pi_b^T(x, \delta) P \pi_b(x, \delta), \pi_b = \begin{bmatrix} x \\ \pi \end{bmatrix} \quad (3)$$

where $P \in \mathbb{R}^{m \times m}$ is a *constant symmetric* matrix, not necessarily positive definite, and $\pi: \mathbb{R}^N \times \mathcal{D} \mapsto \mathbb{R}^p$ is mapping, in which each element is a monomial in (x, δ) , or a smooth rational function with a monomial numerator. The arguments of π and π_b will be suppressed below.

Applications of the linear fractional transformation (LFT) and further algebraic steps have been proposed [5,6] to transform the system equation $\dot{x} = f(x, \delta) = A(x, \delta)x$ into the desired differential-algebraic representation that was introduced in the same references:

$$\begin{aligned} \dot{x} &= A(x, \delta)x = Ax + B\pi \quad x_0 \in \mathcal{X} \\ 0 &= \mathcal{N}_{\pi_b}(x, \delta)\pi_b \quad \delta \in \mathcal{D}, \dot{\delta} \in \tilde{\mathcal{D}} \end{aligned} \quad (4)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times p}$ are constant matrices, and $\mathcal{N}_{\pi_b}(x, \delta) \in \mathbb{R}^{q \times m}$ is an *affine* matrix function in (x, δ) also known as an “annihilator”. These transformations also result in the dimension reduction of the optimization problem compared to the results presented in Ref. [4]. The representation in Eq.(4) separates the linear part of the system (x) from its nonlinear part (π).

3. The SDP Problem for Estimating the DOA

In this section, how to construct the semi-definite optimization task (SDP) is presented, which will uniquely characterize an appropriate Lyapunov function and its maximal invariant level set ϵ_α as an estimate of the true domain of attraction. It is assumed that polytopes \mathcal{X}, \mathcal{Y} , and \mathcal{D} are already given.

Using the model representation of Eq.(4) and a Lyapunov function candidate of the form in Eq.(3), sufficient LMIs for the Lyapunov conditions in Eq.(2) can be formulated. According to Finsler’s lemma (Lemma 2.1 in Ref. [4]), if real-valued matrices $L_b \in \mathbb{R}^{m \times q}$ and $L_a \in \mathbb{R}^{(n+2p+n^2+np) \times (2q+n^2+nq)}$ exist

$$\begin{aligned} \forall (x, \delta) \in \vartheta(\mathcal{X} \times \mathcal{D}): \\ P + L_b \mathcal{N}_{\pi_b}(x, \delta) + \mathcal{N}_{\pi_b}^T(x, \delta) L_b^T > 0 \end{aligned} \quad (5)$$

$$\begin{aligned} \forall (x, \delta, \dot{\delta}) \in \vartheta(\mathcal{X} \times \mathcal{D} \times \tilde{\mathcal{D}}): \\ P_a + P_a^T + L_a \mathcal{N}_{\pi_a}(x, \delta, \dot{\delta}) + \mathcal{N}_{\pi_a}^T(x, \delta, \dot{\delta}) < 0 \end{aligned} \quad (6)$$

then the conditions of Eq.(2) are satisfied for every $(x, \delta, \dot{\delta}) \in \mathcal{X} \times \mathcal{D} \times \tilde{\mathcal{D}}$. Variables P_a and $\mathcal{N}_{\pi_a}(x, \delta, \dot{\delta})$ are

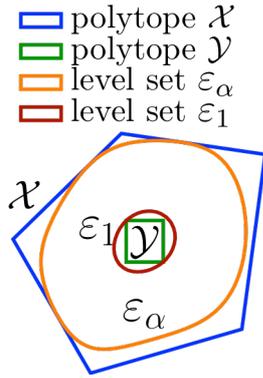


Figure 1. Illustration of the conditions regarding \mathcal{X} and \mathcal{Y} . Inside polytope \mathcal{X} of the Lyapunov conditions in Eq.(2) are required. The α -level set ε_α of the Lyapunov function should lie inside polytope \mathcal{X} (hence it is invariant). The 1-level set ε_1 should be around polytope \mathcal{Y} . This condition ensures that the problem has a bounded solution.

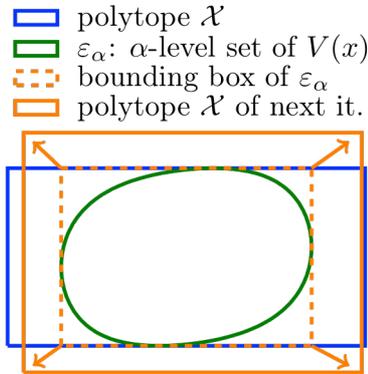


Figure 2. Polytope \mathcal{X} is evaluated through iterations considering the bounding box of the obtained α -level set.

defined in Eqs.(3.16) and (3.22) of Ref. [15], respectively. $\vartheta(\cdot)$ denotes the corner points (vertices) of a given polytope.

Since the Lyapunov conditions are satisfied within \mathcal{X} , an attempt is made to identify the maximal α -level set

$$\varepsilon_\alpha = \{x \in \mathcal{X} | V(x) = \alpha, 1 \leq \alpha\},$$

which lies inside \mathcal{X} . That level set will be *invariant*, in the sense that every trajectory entering this region will never leave it.

$\alpha \geq 1$ is defined, as a free variable of the optimization task, and ε_α is constrained inside \mathcal{X} . One can observe that by maximizing α an unbounded feasible solution is obtained, as the function $V(x, \delta)$ can be scaled arbitrarily. Therefore, as Fig.1 illustrates, an auxiliary polytope \mathcal{Y} is defined around the locally stable origin inside \mathcal{X} , constraining the 1-level set $\varepsilon_1 = \{x \in \mathcal{X} | V(x) = 1\}$ to be around \mathcal{Y} .

According to Finsler's lemma, for every $k = \overline{1, M_X}$ and $l = \overline{1, M_Y}$, if real-valued matrices $L_{C_k}, \bar{L}_{C_l} \in \mathbb{R}^{m \times q}$, $M_{C_k}, \bar{M}_{C_l} \in \mathbb{R}^{(m+1) \times n}$ exist

Table 1. Number and dimensions of parameter independent LMIs of the optimization task.

	Dimension of the LMIs	No. of LMIs
Eq.(5)	m	$M_X \cdot M_D$
Eq.(6)	$n + 2p + n^2 + np$	$M_X \cdot M_D \cdot M_{\bar{D}}$
Eq.(7)	$m + 1$	$2M_X \cdot M_D$
Eq.(8)	$m + 1$	$2M_Y \cdot M_D$

Table 2. Free variables of the optimization task and the number of (scalar) symbolic decision variables they introduce into the optimization task.

Matrix variable	number of (scalar) independent decision variables appearing in the SDP
α	1
P	$\frac{1}{2} m(m+1)$
L_b	mq
L_a	$(n + 2p + n^2 + np)(2q + n^2 + nq)$
L_{C_k} and \bar{L}_{C_l}	$(M_X + M_Y) \times mq$
M_{C_k} and \bar{M}_{C_l}	$(M_X + M_Y) \times (m + 1)n$
The dimensional parameters are the following:	
n	number of state variables (size of x)
P	number of elements in π
m	number of elements in π_b , $(n + p)$
q	number of rows in annihilator $\mathcal{N}_{\pi_b}(x, \delta)$
M_X	number of corner points of \mathcal{X}
M_Y	number of corner points of \mathcal{Y}
M_D	number of corner points of \mathcal{D}
$M_{\bar{D}}$	number of corner points of $\bar{\mathcal{D}}$

$$Q_k^T P_{C_k}^{(\omega)}(x, \delta) Q_k \geq 0 \quad \forall (x, \delta) \in \vartheta(\mathcal{F}_k^X \times \mathcal{D}) \quad (7)$$

$$\bar{Q}_k^T \bar{P}_{C_k}^{(1)}(x, \delta) \bar{Q}_k \leq 0 \quad \forall (x, \delta) \in \vartheta(\mathcal{F}_k^Y \times \mathcal{D}) \quad (8)$$

then ε_α is inside \mathcal{X} and ε_1 is around \mathcal{Y} . Variables $Q_k, \bar{Q}_k, \bar{P}_{C_k}^{(1)}$, and $P_{C_k}^{(\alpha)}$ are defined in Eqs.(3.34) and (3.36) of Ref. [15], \mathcal{F}_k^X denotes the k^{th} facet of \mathcal{X} , furthermore, M_X denotes the number of facets of \mathcal{X} .

The LMI conditions of Eqs.(5)-(8) are affine parameter-dependent LMIs, which can be computationally handled by checking their feasibility at the corner points of the polytopic region, on which the parameters (x, δ, δ) are defined. Depending on the number of the corner points, a given number of parameter independent LMI conditions (Table 1) is obtained.

The SDP task can be summarized as follows. In order to find the maximal invariant level set ε_α of the Lyapunov function, one should maximize α , under the following conditions:

$$\begin{aligned} \text{Eq.(5)} & \quad V(x, \delta) \text{ is positive on } \mathcal{X} \times \mathcal{D} \\ \text{Eq.(6)} & \quad \dot{V}(x, \delta, \delta) \text{ is negative on } \mathcal{X} \times \mathcal{D} \times \bar{\mathcal{D}} \\ \text{Eq.(7)} & \quad \varepsilon_\alpha \text{ lies inside } \mathcal{X} \\ \text{Eq.(8)} & \quad \mathcal{Y} \text{ is inside } \varepsilon_1 \end{aligned}$$

The number of LMIs and their dimensions are given in Table 1, furthermore, the free variables of the optimization task are listed in Table 2.

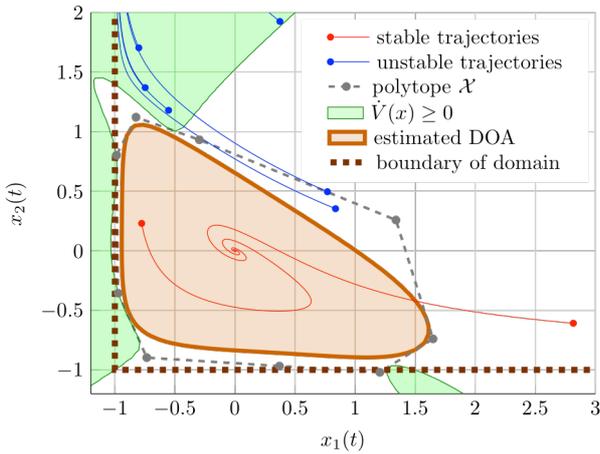


Figure 3. Phase diagram of the 2D locally stable system. Red trajectories converge at the origin, the blue ones do not converge at the origin. The estimated DOA and the used polytope \mathcal{X} are illustrated by the orange-filled region and by the grey-dashed polytope, respectively. The green area highlights the subset of the state space where the Lyapunov function features a positive time-derivative.

3.1. Finding the Most Appropriate Outer Polytope

In the case of 2D systems, polytope \mathcal{X} is evaluated manually through iterations. When having a higher-dimensional system ($n \geq 3$), the iterative procedure [6] is applied that starts from an initial polytope $\mathcal{X}^{(0)}$, then a new polytope $\mathcal{X}^{(1)}$ is defined by enlarging the axis aligned bounding box of the obtained α -level set $\varepsilon_{\alpha}^{(0)}$. During the iterations, $\mathcal{X}^{(i)}$ is constrained to be a rectangular polytope. Fig.2 illustrates the operation of a single iteration step.

4. Lotka-Volterra Systems

The N -dimensional LV equation has the form

$$\dot{x} = \text{diag}(\bar{x} + b), \quad A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n, \quad (9)$$

where $\text{diag}(a)$, $a \in \mathbb{R}^n$ denotes an $n \times n$ square diagonal matrix with the x_1, \dots, x_n on the main diagonal. The system is translated into its, by assumption, unique interior equilibrium point $x^* = -A^{-1}b$ by introducing the centred state vector $x = \bar{x} - x^*$. Then, an autonomous nonlinear system of the form $\dot{x} = \mathcal{A}(x)x$ is obtained, where the matrix function \mathcal{A} can be expressed as: $\mathcal{A}(x) = \text{diag}(x + x^*)A$. By applying the LFT and the further algebraic model transformation steps, a model is obtained in the representation of Eq.(4), where the entries of $\pi(x)$ are second order monomials of the state variables.

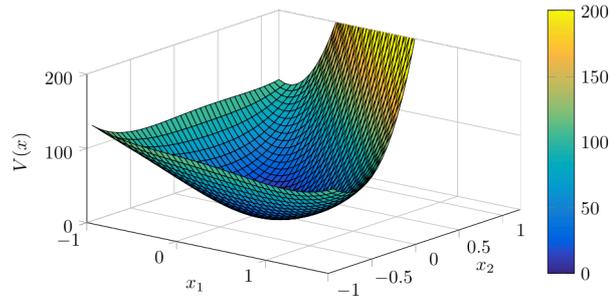


Figure 4. The obtained Lyapunov function in the case of the 2D LV system.

5. Numerical Results

In this section, the applicability of the approach presented above is illustrated through different locally stable Lotka-Volterra models. The results presented here have been computed in a MATLAB environment. For symbolic computations, MATLAB'S built-in Symbolic Math Toolbox was used based on MuPAD. For linear fractional transformations (LFT), the Enhanced LFR-toolbox is used [16, 17]. To model and solve semi-definite optimization (SDP) problems, Mosek solver with YALMIP was used [18].

5.1. 2D Lotka-Volterra System

A locally stable LV system is considered with the model matrix:

$$A = \begin{bmatrix} -2 & -3 \\ 1.4 & 1 \end{bmatrix},$$

with a unique interior equilibrium point $x^* = [1 \ 1]^T$. In Fig.3, some trajectories of the centred system can be observed from different initial conditions. The red trajectories converge at the equilibrium point, the blue ones do not tend to the equilibrium point. After solving the corresponding SDP, the obtained Lyapunov function is $V(x) = \pi_b^T P \pi_b$ (Fig.4), where

$$\pi = [x_1^2 \quad x_1 x_2 \quad x_2^2]^T, \pi_b = \begin{bmatrix} x \\ \pi \end{bmatrix} \quad (10)$$

$$P = \begin{bmatrix} 87.39 & 63.83 & -3.45 & 0 & 0 \\ 63.83 & 165.26 & 67.24 & 100.85 & 28.76 \\ -3.45 & 67.24 & 7.63 & 0.15 & 0 \\ 0 & 100.85 & 0.15 & 38.78 & 39.97 \\ 0 & 28.76 & 0 & 39.97 & 15.53 \end{bmatrix}$$

In Fig.3, the shape of the manually chosen polytope \mathcal{X} , and the obtained invariant α -level set of the Lyapunov function can be seen, which is considered as the estimated DOA (filled orange region). Areas can be observed, where the Lyapunov function's time derivative is positive (green region) can also be observed to be completely outside of \mathcal{X} . Fig.5 illustrates how the value of the Lyapunov function decreases along the trajectories.

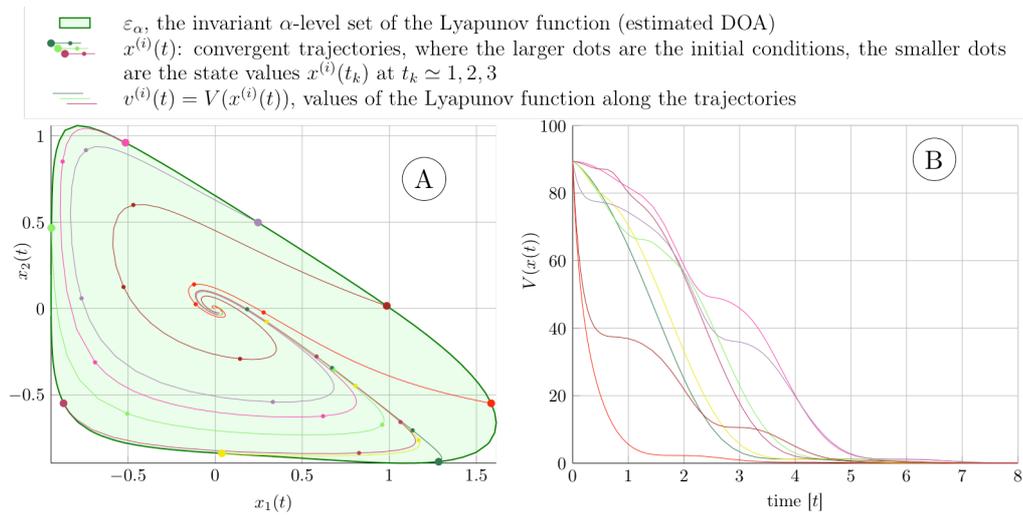


Figure 5. In Panel A, the obtained invariant level set is shown, furthermore some trajectories $x^{(i)}(t)$ of the system can be seen in different colors with initial conditions $x_0^{(i)}$ close to the boundary of the estimated DOA. The values of the Lyapunov function along the trajectories ($v^{(i)}(t)$) are presented in Panel B, respectively.

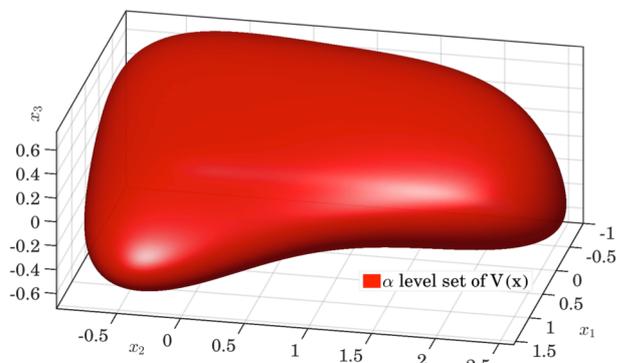


Figure 6. Estimated DOA of the locally stable 3D Lotka-Volterra system (red mesh) and the corresponding final polytope \mathcal{X} .

5.2. 3D Lotka-Volterra System

A locally stable 3D LV system has been chosen with the model matrix

$$A = \begin{bmatrix} 0.06 & 0.21 & 0.83 \\ -2.47 & -2.10 & -3.64 \\ 0.06 & 0.47 & -0.45 \end{bmatrix}.$$

Similarly to the 2D example, the equilibrium point is set to $x^* = [1 \ 1 \ 1]^T$. The monomial set appearing in π_b contains the state variables and every possible 2nd order monomial:

$$\pi^T = [x_1^2 \ x_1x_2 \ x_1x_3 \ x_2^2 \ x_2x_3 \ x_3^2] \quad (12)$$

Matrix P is a 9x9 symmetric matrix. Fig.6 illustrates the invariant α -level set of the obtained Lyapunov function (3D red volume). In Fig.7, one can see the cross-sections of the 3D invariant region from three different viewpoints. Some trajectories of the system are shown in red (stable) and blue (unstable).

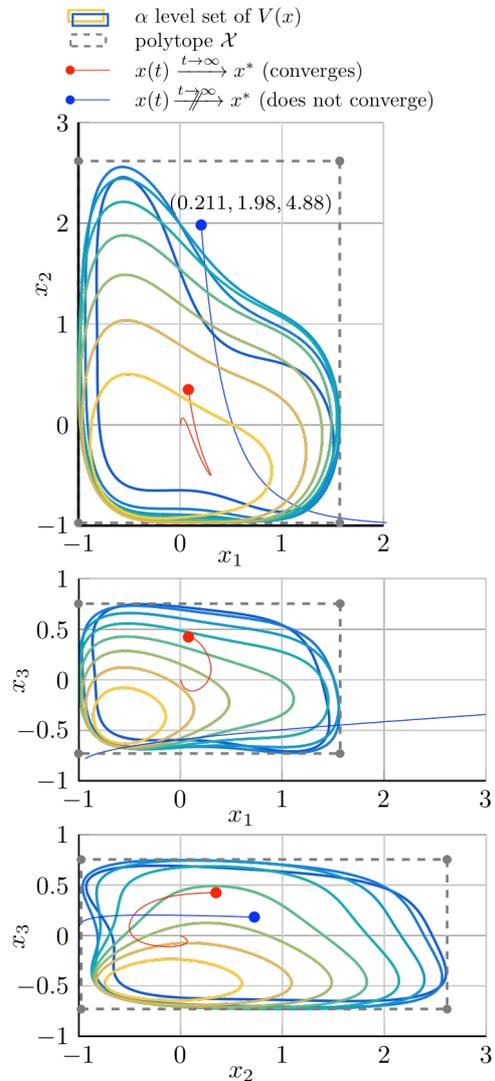


Figure 7. Cross-sectional view of the 3D estimate along the major planes x_i, x_j and using different values for x_k , where (i, j, k) is a permutation of $(1, 2, 3)$.

Table 3. Number and sizes of the LMIs in the case of N -dimensional LV systems.

size	number	description
$m_b \times m_b$	2^n	positivity of $V(x)$
$m_a \times m_a$	2^n	negativity of $\dot{V}(x)$
$m_b \times m_b$	$n \cdot 2^n$	$\varepsilon_\alpha \in \mathfrak{Int}(\mathcal{X})$
$m_b \times m_b$	$n \cdot 2^n$	$\mathcal{Y} \in \mathfrak{Int}(\varepsilon_\alpha)$

m_b is the size of the Lyapunov matrix P , $m_a = \frac{1}{2}(n^3 + 5n^2 + 4n)$ is the size of the matrix $P_s(P)$, which appears in the LMI ensuring the negativity of the time derivative of the Lyapunov function.

5.3. 4D Lotka-Volterra System

The method was applied to a locally stable 4D LV system with a unique interior equilibrium point:

$$x^* = [1 \ 1 \ 1 \ 1]^T. \tag{13}$$

The model matrix of the LV system was chosen to be:

$$A = \begin{bmatrix} -4.7126 & -1.5833 & 1.5346 & 2.3230 \\ -9.2461 & -3.1634 & 2.8648 & 2.7796 \\ -14.258 & -4.5477 & 3.6104 & 4.6238 \\ -3.1687 & -0.8016 & 1.2287 & 1.2656 \end{bmatrix}.$$

Due to the fact that model matrix A is a “full” matrix, i.e. there are no zero entries in it, every second-order monomial will appear in the equation of the system. As a consequence, the LFT will produce a model Eq.(4), in which π will contain every possible

second-order monomial. This means that the number of monomials (including the state variables) is $m_b = (n^2 + 3n) / 2$ in the case of a “full” model matrix.

In Table 3, the number and sizes of the LMIs have been summarised in the case of N -dimensional LV systems with a “full” model matrix. The exponential factor 2^n in Table 3 originates from the rectangular shape of polytope \mathcal{X} possessing 2^n corner points. On the other hand, the feasibility of LMI conditions in Eqs.(7) and (8) should be checked at each corner point of every facet of the polytope. Furthermore, an N -dimensional rectangular polytope has $2n$ facets with 2^{n-1} corner points. As can be seen, a rectangular polytope introduces an exponential increase in the dimension of the given problem. However, N -dimensional intervals can be easily handled compared to arbitrarily shaped polytopes defined by (hyper)-triangle meshes. Fig.8 illustrates the cross sections of the invariant domain along the different pair of axes.

5.4. 2D Uncertain LV System with an Uncertain Equilibrium Point

In this section, the same 2-dimensional system is presented that appeared in Section 5.1 with the same model parameters but with an additional uncertain term $K\delta$, where $K = [1 \ 1]^T$. It is assumed, that $\delta \in \mathcal{D} = [-0.1, 0.1]$ is an unknown constant parameter ($\dot{\delta} = 0$). The equation of the uncertain model is the following:

$$\dot{x} = \text{diag}(\bar{x})(A\bar{x} + K\delta + b) \tag{15}$$

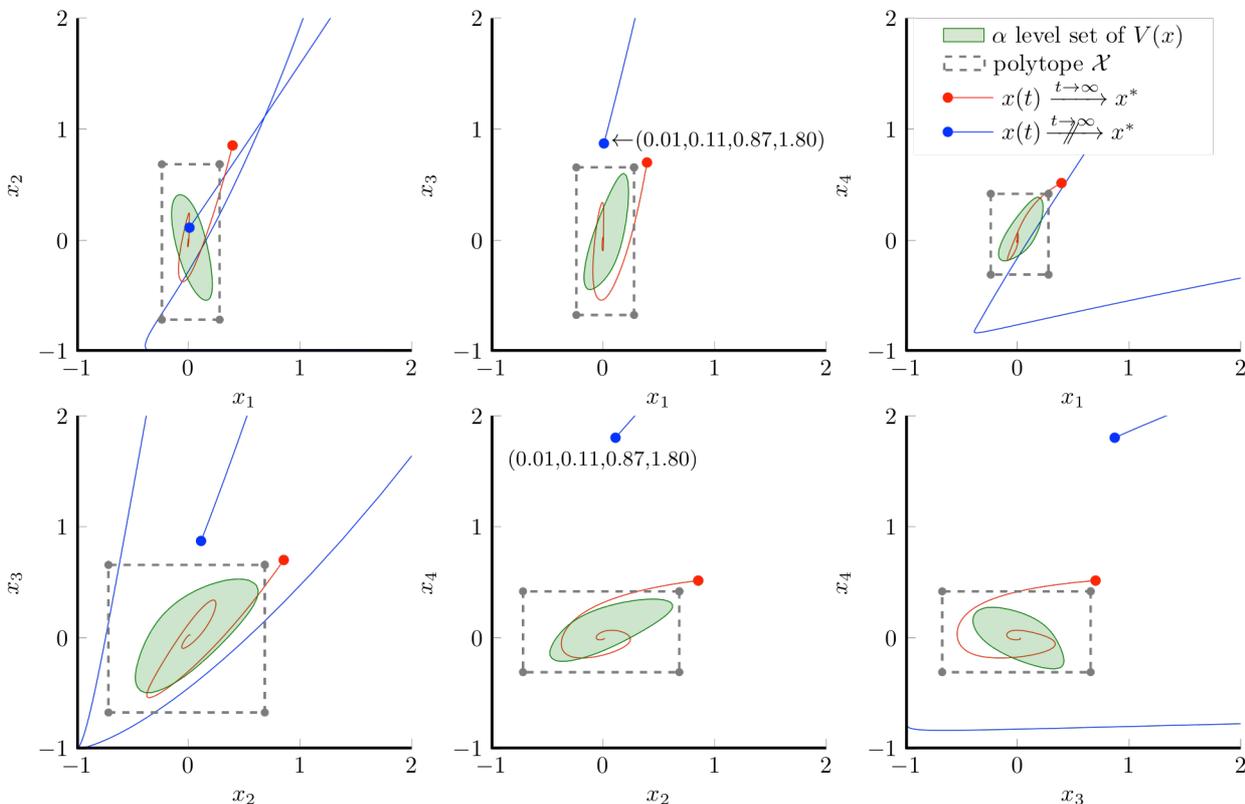


Figure 8. Cross-sectional view of the DOA estimate of the 4D LV system.

The equilibrium point of the uncertain system is an affine function of the uncertain parameter δ :

$$x^*(\delta) = A^{-1}(-K\delta - b) = -A^{-1}K\delta + x_0^*, \quad (16)$$

where x_0^* is the equilibrium point of the system presented in Section 5.1, i.e. when $\delta = 0$. Eq.(15) is translated into the following form:

$$\dot{\bar{x}} = \text{diag}(\bar{x})A(\bar{x} - x^*(\delta)). \quad (17)$$

The centred model of the system around the uncertain equilibrium point $x^*(\delta)$ can be calculated if the new state vector $x = \bar{x} - x^*(\delta)$ is introduced. The equation of the centred system is the following:

$$\dot{x} = \text{diag}(x + x^*(\delta))Ax. \quad (18)$$

If the Lyapunov function depends on δ , especially when possessing an uncertain equilibrium point, it is not straightforward to determine an invariant region by considering the α -level set of the Lyapunov function. However, it is possible to compute a region \mathfrak{I} , which is “invariant with respect to” a larger region \mathfrak{U} ($\mathfrak{I} \subset \mathfrak{U}$), in the sense that every trajectory with an initial condition from \mathfrak{I} will not leave \mathfrak{U} .

In order to compute \mathfrak{I} and \mathfrak{U} , the α -level set $\bar{\varepsilon}_\alpha(\delta)$ of the obtained Lyapunov function $V(x, \delta)$ is determined in the coordinates system of the original system and at the corresponding equilibrium point for every $\delta \in \mathcal{D}$. The overlining in the notation $\bar{\varepsilon}_\alpha(\delta)$ means that this level set is translated into the original coordinates system. As Fig.9 illustrates, the intersection \mathfrak{I} and the union \mathfrak{U} of the α -level sets obtained for different values of δ have been computed. It can be stated that every trajectory starting from region \mathfrak{I} will not leave region \mathfrak{U} .

6. Conclusion

In this work, the Lyapunov function of N -dimensional LV systems ($N = 2, 3, 4$) was successfully computed by using the improved optimization-based method [6, 15]. In the case of each deterministic system, an invariant region was given, as the estimated DOA. In the case of an uncertain LV system with an uncertain interior equilibrium point, two regions ($\mathfrak{I} \subset \mathfrak{U}$) were given, which describe the stable regions of the system.

SYMBOLS

- \dot{x} denotes the time-derivative of function $x(t)$
- \mathbf{L}^T denotes the transpose of a matrix \mathbf{L}
- $\vartheta(\cdot)$ denotes the corner points of a polytope
- $\text{diag}(\mathbf{x})$ denotes the diagonal matrix
- $x = x(t) \in \mathbb{R}^n$ represents the state variables of a dynamical system

-  $\bar{\varepsilon}_\alpha(\delta_i)$: α -level set of $V(x, \delta)$ for different values of δ in the original coordinates system
-  $\bar{x}^*(\delta_i)$: equilibrium point for different values of δ
-  \mathfrak{I} : intersection of $\bar{\varepsilon}_\alpha(\delta) \forall \delta \in \mathcal{D}$
-  \mathfrak{U} : union of $\bar{\varepsilon}_\alpha(\delta) \forall \delta \in \mathcal{D}$

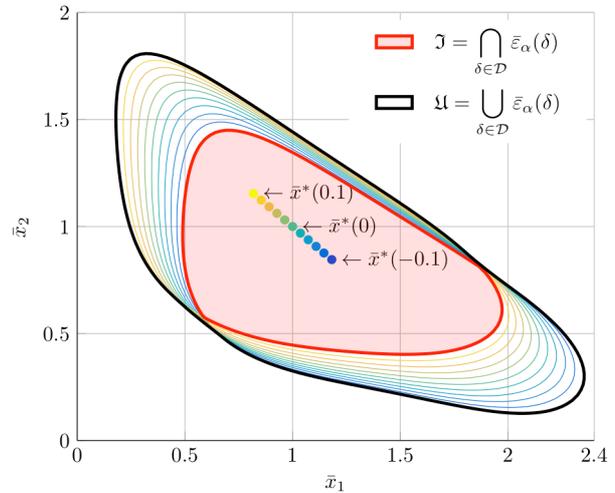


Figure 9. α -level set $\varepsilon_\alpha(\delta_i)$ of the Lyapunov function in the original coordinates system for some values δ_i of the uncertain parameter and the corresponding equilibrium point $x^*(\delta_i)$ of the same color. The red-filled region is the intersection of the level sets for every possible value of δ . The black line shows the contour of the union of the level sets.

- $\delta \in \mathbb{R}^n$ vector of uncertain parameters
- $\pi \in \mathbb{R}^n$ vector valued function, which represents the set of monomial and rational nonlinear terms to be considered in the Lyapunov function
- $\pi \in \mathbb{R}^{m=n+p}$ auxiliary variable denoting $\begin{bmatrix} x \\ \pi \end{bmatrix}$
- $V(x)$ denotes the Lyapunov function
- $\mathcal{X} \subset \mathbb{R}^n, \mathcal{D}, \mathcal{D} \subset \mathbb{R}^d$ bounded polytopes

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