

## PROBABILITY AND EXPECTED TIME OF EMPTYING OF INTERMEDIATE STORAGE UNDER STOCHASTIC OPERATION

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A mathematical model for the operation of an intermediate storage under stochastic operational conditions is presented. The material input of the storage occurs at randomly, both in terms of time and amount of material, while the output is of constant volumetric rate. The main practical question is the required initial amount of material in the storage allowing no emptying of storage. In order to determine that we define a two variable function. We develop an equation satisfied by this function and present solution for the case of Erlang( $n$ ) distributed inter-arrival time. We provide numerical examples and apply this function for solving the original problem.

**Keywords:** Process system, random operation conditions, intermediate storage, reliability, integro-differential equation.

### Introduction

Intermediate storage is frequently used in processing systems. In chemical engineering, the material produced by batch operational units is often collected in a storage system from which it is withdrawn for further processing in the subsequent plant units. The reasons of applying storage may be the different operational characteristics of the input and output subsystems, or necessity of sparing the processed material in case of failures during the operation. It seems to be an important question to determine the necessary size of the storage in order to avoid the overflow and the necessary initial amount of material not to run out the material stored. This is to ensure the work without failure of the continuously operated subsystem supplied by material from the storage system.

Investigating the operation of intermediate storage, researchers realised that the operation is rather stochastic than deterministic [2]. General stochastic model was set up for batch-batch systems [3], and a method was given for determining the size and the initial amount of material required for operation of the system without failures at a given reliability level.

In this paper we investigate such a model in which the input of the storage system is formed by a processing system of a large set of batch operational units while the output of the storage occurs continuously, i.e. the material from the storage is withdrawn at a constant rate. In previous work [4], we investigated this problem assuming Poisson input process. We have set up the equations modelling the behaviour of the system which were solved analytically in some special cases, and we have given a method for solving the sizing and initial

amount problem based on these solutions or computer simulation.

In this paper we deal with the problem of initial amount of material under more general conditions. We determine the probability of running out of material in the storage as a function of the initial amount of material. On the other hand it is interesting to also know the expected time of this event. In order to handle these problems together, we introduce a general function which at the value zero provides the probability of running out of material, and its derivative provides the expected time of emptying as well. Applying the tools of probability theory we set up an integro-differential equation satisfied by this function. We show some qualitative properties of the solution and in some cases we determine analytical solutions for it. Furthermore, we present numerical examples of solutions which are applied for solving the practical problem.

### The model: basic assumptions and notations

Consider a processing system consisting of a batch and a continuous subsystem that are connected by means of an intermediate storage system, shown schematically in *Fig. 1*. The material to be processed is filled into the storage by the input sub-systems. The filling happens randomly, both in terms of time and amount of material. The investigation starts at  $t_0 = 0$  and the time intervals between the consecutive filling points are described by random variables  $t_k, 1, 2, 3, \dots$

Therefore, the  $i^{\text{th}}$  filling is at time  $\sum_{k=1}^i t_k$ .

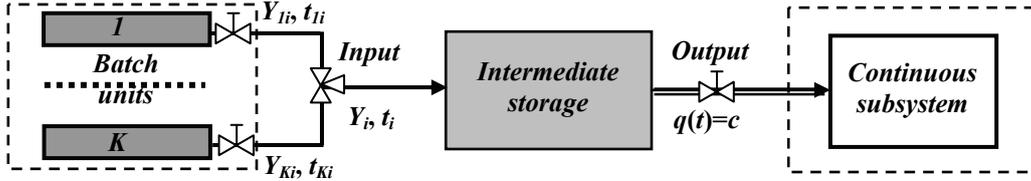


Figure 1: Intermediate storage connecting the batch and continuous subsystems

Let the random variables  $t_i$  be assumed to be nonnegative, independent, identically distributed random variables, hence the input process is a renewal process. If the random variables are exponentially distributed, then the input process is a Poisson process. In a general case let their common distribution function be denoted by  $F(t)$ , while the density function be denoted by  $f(t)$ . Let  $\mu_f$  be their finite expectation.

Let  $N(t)$  denote the (random) number of fillings in the time interval  $[0, t]$ . In the  $i^{\text{th}}$  filling event, the amount of material filled into the storage is random as well, and it is described by the nonnegative random variable  $Y_i$ ,  $i = 1, 2, \dots$ . These are supposed to be independent of each other. Their distribution function is denoted by  $G(y)$ , the density function is  $g(y)$ , while the finite expectation is  $\mu_g$ . Furthermore, we assume that the input process  $N(t)$  and  $Y_i$  are independent as well.

The output process is a deterministic process with constant output rate  $c$ . The main goal is to determine the initial amount of material necessary at a given reliability level, which means that there occurs no running out of the material with a given probability. For this purpose it is worth investigating the change of material in storage as a function of time.

The change of material is the difference of the filled and the withdrawn material. If we denote the initial amount of material by  $x$ , which is the amount of material at  $t = 0$ , and we sum up the change to the initial amount of material  $x$ , we get the function describing the amount of material in the storage in terms of time. This is a random function, which means that considering it in terms of time provides a stochastic process. In order not to run out of material it is necessary to satisfy the following inequality

$$V(t) = x + \sum_{i=1}^{N(t)} Y_i - ct \geq 0 \quad (1)$$

for any value of  $t \geq 0$  where  $V(t)$  denotes the amount of the material in the storage at time  $t$ . As  $V(t)$  is a stochastic process, the inequality (1) holds for some realization and does not hold for other ones, i.e. it holds only with a given probability (reliability).

Let us introduce a function describing reliability, i.e. the probability of not emptying as a function of the initial amount of material.

If  $x$  is the initial amount of material then we define

$$R(x) = P\left(\left\{0 \leq x + \sum_{i=1}^{N(t)} Y_i - ct \quad \forall t : 0 \leq t\right\}\right).$$

The probability of running out of the material is given by

$$\psi(x) = 1 - R(x),$$

and the time of emptying is expressed as

$$T_V(x) = \begin{cases} \infty, & \text{if } V(t) \geq 0 \text{ for any } t \geq 0 \\ \inf\{t \geq 0 : V(t) < 0\}, & \text{if there exists } t \geq 0 : V(t) < 0 \end{cases}$$

Furthermore let the function  $\phi(x, \delta)$  be defined as follows. For  $0 \leq x$  and  $0 \leq \delta$  we have

$$\phi(x, \delta) = E(e^{-\delta T_V(x)} 1_{T_V(x) < \infty}).$$

This function is, in essence, the Laplace transform of the probability density function of the finite time of running out of the material. It can be easily shown that

$$\phi(x, \delta) = \psi(x),$$

and

$$-\frac{\partial \phi(x, \delta)}{\partial \delta} \Big|_{\delta=0} = E(e^{-\delta T_V(x)} 1_{T_V(x) < \infty}),$$

hence the function  $\phi(x, \delta)$  is a suitable tool for determining the reliability and the expectation as well. Consequently, we will deal with the function  $\phi(x, \delta)$ .

We mention that similar function was introduced in insurance mathematics to investigate the ruin probability in [1] called Gerber-Shiu discounted penalty function.

### Equations for the function $\phi(x, \delta)$

First we note that the function  $\phi(x, \delta)$  is monotonic decreasing in both variables. It is continuous and the inequality  $0 \leq \phi(x, \delta) \leq 1$  holds. Furthermore it can be proved that:

**Theorem 1:** For any  $x \geq 0$  and  $\delta \geq 0$  function  $\phi(x, \delta)$  satisfies the following integro-differential equation:

$$\phi(x, \delta) = \int_0^x \int_0^c e^{-\delta t} \phi(x + y - ct, \delta) f(t) g(y) dt dy + \int_{\frac{x}{c}}^{\infty} e^{-\delta t} f(t) dt. \quad (2)$$

**Theorem 2:** Equation (2) has a unique solution for  $\delta > 0$  in the set of bounded functions. This solution tends to zero in exponential order. Namely  $\phi(x, \delta) \leq Ke^{-rx}$  for suitable constants  $0 < K$  and  $0 < r$ , depending on  $\delta$ .

In the case of  $\delta = 0$  the bounded solution of Equation (2) is not unique, as the Heaviside function, i.e. constant of value one for all  $x \geq 0$  is a solution to Equation (2)

and, if  $\frac{\mu_g}{\mu_f} > c$  and  $\int_x^\infty f(t)dt \leq K_2 e^{-rx}$ ,  $r > 0$ , then there exists such a solution as well for which  $\phi(x, 0) \leq K_1 e^{-rx}$  with a suitable constant  $K_1 > 0$ .

**Remark:** Condition  $\frac{\mu_g}{\mu_f} > c$  means that during a unit time interval the expectation of the amount of filled material is more than the amount of withdrawn material. In the opposite case it can be proven that the solution of the processing problem is

$$\phi(x, 0) \equiv 1, \quad x \geq 0.$$

Theorem 2 ensures that in the case  $\frac{\mu_g}{\mu_f} > c$  for any

$0 < \alpha < 1$  there exists such an initial amount of material  $x$  for which the material in the intermediate storage will not run out with probability  $1 - \alpha$  hence the processing problem can be solved theoretically. The solution will be provided by taking the inverse function of  $\phi(x, 0) = \psi(x)$  at point  $y = \alpha$ . Consequently we want to determine the solution of Equation (2).

**Theorem 3:** Let the random variables be the sum of  $n$  independent, exponentially distributed random variables with parameter  $\lambda$ , that is Erlang( $n$ ) distributed. Now, one can prove that Equation (2) can be transformed into the following form:

$$\begin{aligned} \sum_{i=0}^n \frac{\partial^i \phi(x, \delta)}{\partial x^i} \left( \frac{\delta + \lambda}{c} \right)^{n-i} \binom{n}{i} &= \\ &= \left( \frac{\lambda}{c} \right)^{n \infty} \int_0^\infty \phi(x + y, \delta) g(y) dy \end{aligned} \quad (3)$$

with initial conditions

$$\left. \frac{\partial^i \phi(x, \delta)}{\partial x^i} \right|_{x=0} = \left( -\frac{\delta}{c} \right)^i \quad i = 0, 1, \dots, n-1 \quad (4)$$

If  $\delta > 0$ , then the function

$$\phi(x, \delta) = \sum_{i=1}^m \sum_{j=0}^{n_i-1} c_{ij}(\delta) x^j e^{-k_i(\delta)x} \quad (5)$$

is the unique solution of Equation (3) where  $k_i(\delta)$  is the root of the equation

$$\left( \frac{\delta + \lambda}{c} - k \right)^n - \left( \frac{\lambda}{c} \right)^{n \infty} \int_0^\infty e^{-ky} g(y) dy = 0, \quad (6)$$

which is the characteristic equation of Equation (3). The multiplicity of  $k_i(\delta)$  is  $n_i$ , and  $\sum_{i=1}^m n_i = n$ . The constant

values  $c_{ij}(\delta)$  can be uniquely determined from (4) by solving a system of linear equations. If the limit

$$k_i(0) = \lim_{\delta \rightarrow 0^+} k_i(\delta) \quad (i=1, \dots, m)$$

exists, then

$$\psi(x) = \sum_{i=1}^m \sum_{j=0}^{n_i-1} c_{ij}(0) x^j e^{-k_i(0)x}$$

and

$$c_{ij}(0) = \lim_{\delta \rightarrow 0^+} c_{ij}(\delta).$$

**Corollary:** If the assumptions of Theorem 3 are valid and the roots of characteristic equation are of multiplicity 1, then

$$\phi(x, \delta) = \sum_{i=1}^n c_i(\delta) e^{-k_i(\delta)x}$$

and

$$\psi(x) = \sum_{i=1}^n c_i(0) e^{-k_i(0)x}.$$

Now

$$E(T_V(x) | T_V(x) < \infty) = - \sum_{i=1}^n (c_i'(0) - x k_i'(0) c_i(0)) e^{-k_i(0)x}$$

where

$$\begin{aligned} k_i'(0) &= \\ &= \frac{\frac{n}{c} \left( \frac{\lambda}{c} - k_i(0) \right)^{n-1}}{\left( \frac{\lambda}{c} \right)^{n \infty} \int_0^\infty e^{-k_i(0)y} y g(y) dy - n \left( \frac{\lambda}{c} - k_i(0) \right)^{n-1}} \end{aligned}$$

Especially, if  $n = 1$ , that is the inter-arrival times are exponentially distributed, then

$$\phi(x, \delta) = e^{-k_i(\delta)x} \quad (7)$$

$$\psi(x) = e^{-k_i(0)x} \quad (8)$$

$$E(T_V(x) | T_V(x) < \infty) = \frac{x}{c - \lambda \int_0^\infty e^{-k_i(0)y} y g(y) dy} e^{-k_i(0)x}. \quad (9)$$

If  $Y_k$  is exponentially distributed, that is

$$g(y) = \frac{1}{\mu_g} e^{-\frac{1}{\mu_g} y}, \quad y \geq 0,$$

then

$$\phi(x, \delta) = e^{-\frac{(-\frac{c}{\mu_g} - \lambda - \delta) + \sqrt{(\frac{c}{\mu_g} - \lambda - \delta)^2 + \frac{4c\delta}{\mu_g}}}{2c} x} \quad (10)$$

$$\psi(x) = e^{-\left(\frac{\lambda}{c} - \frac{1}{\mu_g}\right)x} \quad (11)$$

$$E(T_V(x) | T_V < \infty) = \frac{\lambda \mu_g x}{c(\lambda \mu_g - c)} e^{-\left(\frac{\lambda}{c} - \frac{1}{\mu_g}\right)x} \quad (12)$$

In the case when  $g(y)$  is a Dirac-delta function at  $y = 1$ , that is  $Y_k \equiv 1$ , then an equation corresponding to Equation(2) can be set up by similar argumentation and its solution expressed as

$$\phi(x, \delta) = e^{-k_1(\delta)x} \quad (13)$$

$$\psi(x) = e^{-k_1(0)x} \quad (14)$$

$$E(T_V(x) | T_V < \infty) = \frac{x}{c(1 + k_1(0)) - \lambda} e^{-k_1(0)x}, \quad (15)$$

where  $k_1(\delta)$  is the unique root with positive real part of the characteristic equation

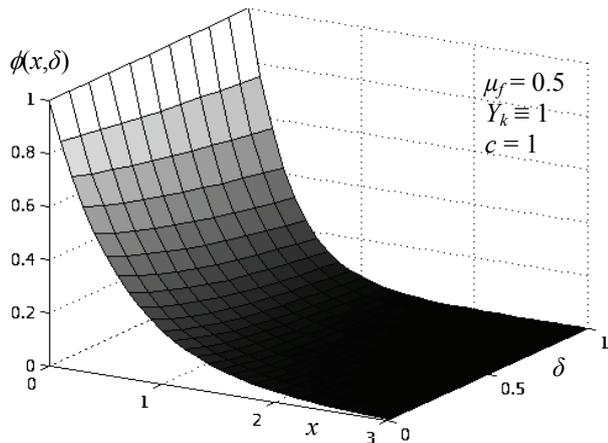
$$\frac{\delta + \lambda}{c} - k - \frac{\lambda}{c} e^{-k} = 0 \quad (16)$$

for  $\delta \geq 0$ .

We note that as  $\psi(x)$  is exponential function in both cases, consequently its inverse function can be analytically determined easily.

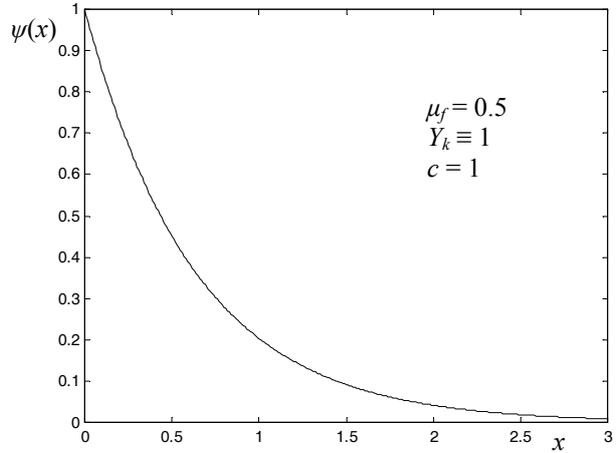
### Numerical results

We discuss some examples of functions presented in the previous sections. *Fig. 2* shows the function  $\phi(x, \delta)$  determined by expression (13) in the case of exponentially distributed time interval  $t_k$  with expected value  $\mu_f = 0.5$  ( $\lambda = 2$ ), constant amount of filled material  $Y_k = 1$  and constant withdrawing rate  $c = 1$ . Here,  $x$  is the initial amount of material and  $\delta$  is a parameter. The exponent  $k_1(\delta)$  was computed numerically from formula (16).



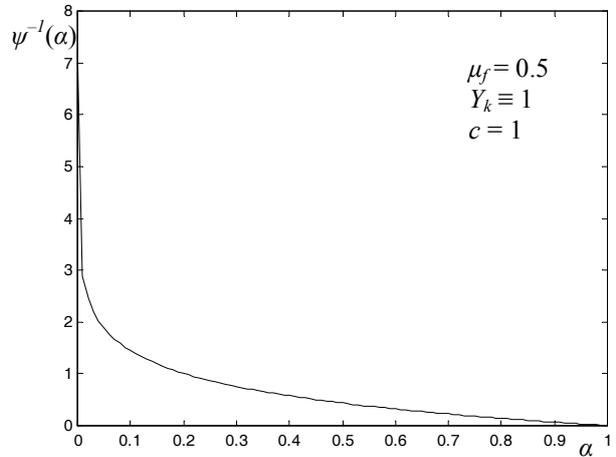
*Figure 2:* The form of function  $\phi(x, \delta)$  depending on the initial amount of material  $x$  and parameter  $\delta$

*Fig. 3* demonstrates  $\phi(x, 0) = \psi(x)$  using the previously given parameters. The exponent in expression (14) is actually 1.5936.



*Figure 3:* Diagram of function  $\psi(x)$  describing the probability of running out of the material depending on the initial amount of material

The inverse function of  $\psi(x)$ , i.e.  $\psi^{-1}(\alpha)$ , provides the necessary initial amount of material in the function of variable  $\alpha$ , where  $\alpha$  provides the probability of running out of material in storage. Function  $\psi^{-1}(\alpha)$  is shown in *Fig. 4* depending on the parameter  $\alpha$ .



*Figure 4:* The initial amount of material as a function of the probability of emptying the storage

The expectation of the time of emptying given by formula (15) as a function of the initial amount of material can be seen in *Fig. 5*.

The function  $\phi(x, \delta)$  given by expression (10) is shown in *Fig. 6* in the case of exponentially distributed consecutive filling times with parameter  $\mu_f = 0.5$  and exponentially distributed filled amount of material with parameter  $\mu_g = 1$ .

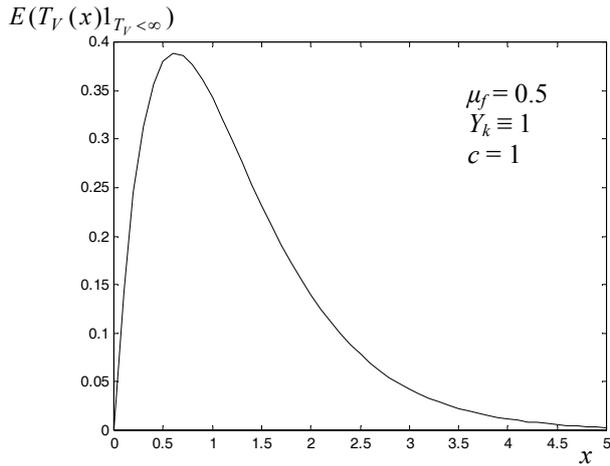


Figure 5: The expectation of the time of emptying the storage depending on the initial amount of material

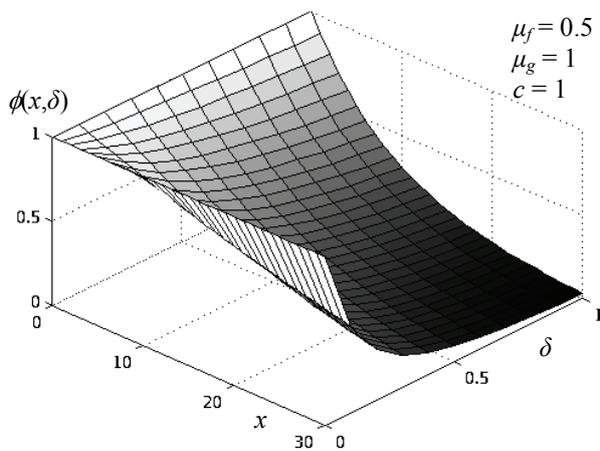


Figure 6: Function  $\phi(x, \delta)$  in the case of Poisson filling process and exponentially distributed filled material into storage

The functions expressing the probability of emptying can be seen in Fig. 7 using different values for the expectation of the filled material.

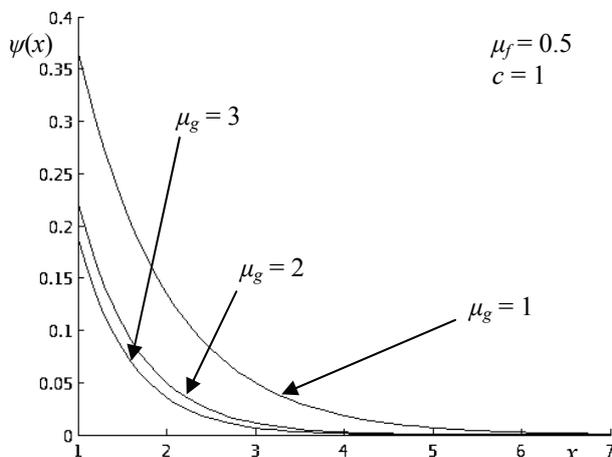


Figure 7: The probability of emptying storage  $\psi(x)$  as a function of initial amount of material

In Fig. 8 we represent the function  $\phi(x, \delta)$  in case of Erlang(2) distributed consecutive input times with  $\lambda = 0.3$ , that is  $E(t_i) = \frac{2}{0.3}$ , and lognormal distribution of the filled amount of material. Parameter  $m$  equals 2, the standard deviation of the normal distribution is  $\sigma = 1$ , hence the expectation of the lognormal distribution is  $E(Y_i) = e^{m+\sigma^2/2} = 12.18$ . The solution of the integro-differential equation (3) is given by expression (5). The exponents were determined numerically from the equation (6). The integral for the case of lognormal distribution of the filled amount of material, i.e. the Laplace transform of the lognormal density function can not be computed analytically. It was determined numerically by using Gauss-Laguerre quadrature formulas based on polynomial of degree 20. For this case, the bivariate function  $\phi(x, \delta)$  is presented in Fig. 8.

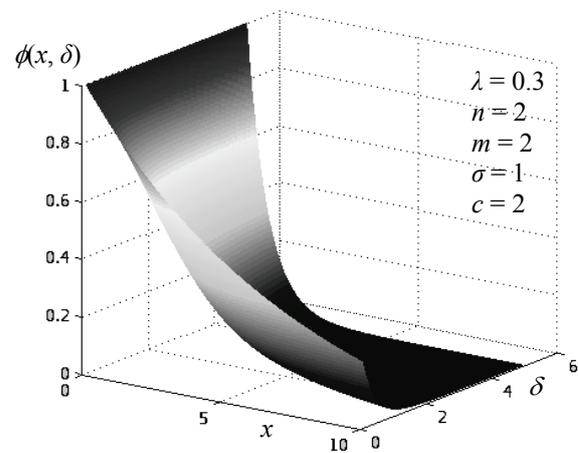


Figure 8: Function  $\phi(x, \delta)$  in case of Erlang (2) distributed inter-arrival times and lognormal distribution of filled amount of material

Finally, let us see an example in details how to determine the initial amount of material according to the reliability level  $1 - \alpha = 0.95$ , i.e. we would like to determine how much initial amount of material is needed if we want to assure the operation of the system without running out of material with probability 0.95. It means that the initial amount of material in the storage  $x$  is to be determined corresponding to  $R(x) = 1 - \alpha = 0.95$ .

Let us suppose that the distribution of the consecutive filling times is Erlang(2) distribution with parameters  $\lambda = 2.1$ ,  $c = 1$  and  $Y_i \equiv 1$ .

First we determine the roots of the equation  $\psi(x) = \alpha$ . As  $\psi(x) = \phi(x, 0)$ , we deal with the function  $\phi(x, \delta)$ , which is of the form (5). Now  $g(y)$  is a Dirac-delta function at  $y = 1$ , hence Equation (6) takes the form

$$\left(\frac{\delta + \lambda}{c} - k\right)^2 - \left(\frac{\lambda}{c}\right)^2 e^{-ky} = 0.$$

This equation has two positive roots, one lies in the interval  $(0, \lambda + \delta)$ , and the another one is in the interval  $(\lambda + \delta, \infty)$  with multiplicities equal 1. These roots were

determined numerically by the Newton method. The solution is of form

$$\phi(x, \delta) = c_1(\delta)e^{-k_1(\delta)} + c_2(\delta)e^{-k_2(\delta)}$$

where the constants  $c_1(\delta)$  and  $c_2(\delta)$  are expressed from the initial conditions for  $i = 0$  and  $i = 1$ , that is from the system of linear equations  $c_1(\delta) + c_2(\delta) = 1$  and  $-k_1(\delta)c_1(\delta) - k_2(\delta)c_2(\delta) = -\frac{\delta}{c}$ . The function  $\phi(x, \delta)$  is shown in Fig. 9.

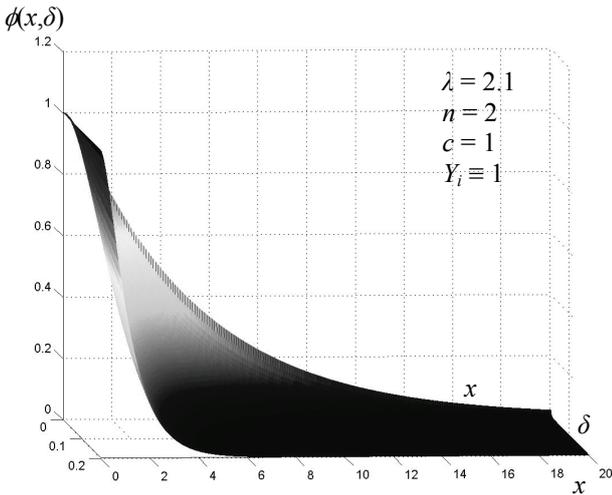


Figure 9: Function  $\phi(x, \delta)$  in the case of Erlang (2) distributed inter-arrival times and constant amount of filled material

The numerical computations provided the values  $k_1(0) = 0.1968$ ,  $k_2(0) = 2.6564$ , while the constants are  $c_1(0) = 1.0800$ ,  $c_2(0) = -0.0800$ , hence finally

$$\psi(x) = 1.0800e^{-0.1968x} - 0.0800e^{-2.6564x}$$

The function  $\psi(x) = 1 - R(x) = \phi(x, 0)$  can be seen in Fig. 10. This function is shown together with the function  $\psi(x)$  equals 0.05 for  $x = 15$ . We numerically determined the roots of equation  $\psi(x) = 0.05$ , which is  $x = 15.61$  and  $\psi(x) = 0.01$  which holds for  $x = 23.7945$ . Therefore, if we want to ensure operation of the continuous processing system without emptying of the

intermediate storage with probability 0.95 we have to start to operate the system with initial amount of material equal to 15.6154 units.

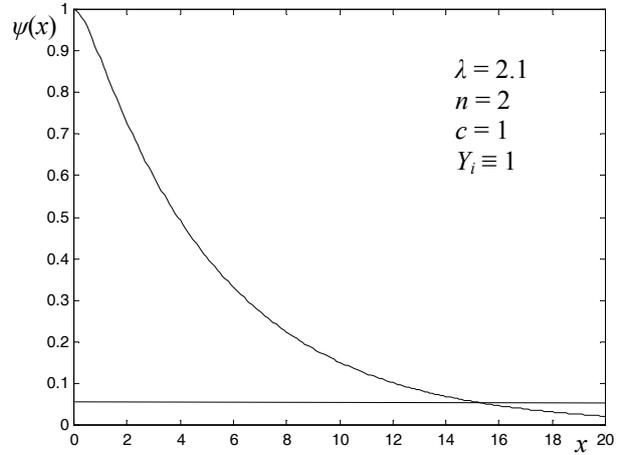


Figure 10: Function  $\psi(x)$  in case of Erlang (2) distributed inter-arrival times and constant amount of material

However, in spite of having this amount of material the storage may still become empty with probability 0.05. The expected time of this event is  $E(T_V | T_V < \infty) = 12.5398$  units.

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