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A COMPARISON OF ADVANCED NON-LINEAR STATE ESTIMATION TECHNIQUES FOR GROUND VEHICLES

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This paper presents a comparative examination of state-of-the-art non-linear state estimation techniques. First, the Sigma-Point (SPKF) stochastic estimators are investigated on the basis of the Unscented Transformation. This is followed by the technique of non-linear symmetry-preserving observers for non-linear dynamics on the basis of symmetrical LIE groups. The variety of the deterministic and stochastic state estimation algorithms is studied the kinematics of ground vehicles. Two cases are described that deal with the reliability of path tracking, the convergence and speed of off-line calibration.

Keywords: state estimation, invariant observers, unscented KALMAN filter, vehicle kinematics

Introduction

Vehicles are indispensable in modern society, and vehicle safety is consequently of importance in everyday life. A lack of information about the state of vehicle and parameters presents a major obstacle for the development of vehicle control systems.

The effectiveness of vehicle stability control largely depends on the accuracy of the vehicle state parameters that are measured by appropriate sensors. Unfortunately, measurements from these sensors contain bias, as well as electrical noise, and can drift with temperature changes. However, the state estimation method is also able to dramatically reduce errors introduced by measurement and process noise contained in the signal.

The focus of this work is the non-linear state estimation techniques. Firstly, the 'Sigma-Point KALMAN filter family' (SPKF) is discussed on the basis of the 'Unscented Transformation' (UT) and STERLING's interpolation [3, 5, 6]. Secondly, the symmetry-preserving observer is presented [15, 16, 18], which is based on Lie-transformation of invariant frames. The various state estimation procedures are then examined and compared through an illustrative non-linear example of vehicle kinematics.

Sigma-Point KALMAN Filters

Sigma-Point KALMAN filter [2, 4, 7] is a collective name for derivativeless KALMAN filters that employs the deterministic sampling based Sigma-point approach to calculate the optimal terms of the state estimation. Common methods use the Scaled Unscented Transformation [4] and STERLING's polynomial interpolation. The former replaces the original set of sigma-points with a transformed set in order to minimize the errors of higher order terms, the latter approximates the derivation of the divided difference filter [10, 11] and central difference filter in [12].

The Scaled Unscented Transformation

The Scaled Unscented Transformation (SUT) has two steps. First, a fixed number of sigma points is chosen to capture the desired moments (at least mean and covariance) of the original distribution. Then the sigma points are propagated through the non-linear function and the moments of the transformed variables are estimated. The advantage of SUT over the Taylor series based approximation is that SUT is better at capturing the higher order moments caused by the non-linear transformation, as discussed in [1, 8-10].

Consider a non-linear function $y = g(x)$ and assume x has mean $\bar{x}$ and covariance P_x. The first two moments of y are calculated by the following two steps:

1) Weighted Sigma-point selection:

$$\begin{aligned} X_0 &= \bar{x} \\ X_i &= \bar{x} + \left(\sqrt{(L+\lambda)P_x}\right)_{i=1\ldots L} \\ X_i &= \bar{x} - \left(\sqrt{(L+\lambda)P_x}\right)_{i=L+1\ldots 2L} \end{aligned} \quad (1)$$

with weights

$$\begin{aligned} w_0^m = \tfrac{\lambda}{L+\lambda} \qquad w_0^c &= \tfrac{\lambda}{L+\lambda} + (1-\alpha^2+\beta) \\ w_i^m = w_i^c &= \tfrac{\lambda}{2(L+\lambda)} \end{aligned} \quad (2)$$

where w_i^m, and w_i^c are associated with the i^{th} sigma-point such that $\sum_{i=0}^{2L} w_i = 1$. Symbols λ, α, and β denote scaling parameters and $\left(\sqrt{(L+\lambda)P_x}\right)_i$ is the i^{th} column of the matrix square root of the weighted $(L+\lambda)P_x$ covariance matrix.

2) Propagation: $Y_i = g(X_i)$

$$\bar{y} \approx \sum_{i=0}^{2L} w_i^m Y_i \qquad P_y \approx \sum_{i=0}^{2L} w_i^c (Y_i - \bar{y})(Y_i - \bar{y})^T$$
$$P_{xy} \approx \sum_{i=0}^{2L} w_i^c (X_i - \bar{x})(Y_i - \bar{y})^T \tag{3}$$

where $\bar{y}$, P_y, and P_{xy} are the approximated mean, covariance and cross-covariance respectively.

The Unscented KALMAN Filter (UKF)

The algorithm of the UKF according to Refs. [3, 7, 8] consists of the following steps:

1) Initialization:

$$\hat{x}_0 = E[x_0],\ P_{x_0} = E\left[(x_0 - x_0)(x_0 - x_0)^T\right] \tag{4}$$

$$\hat{x}_0^a = E\left[\hat{x}_0^a\right] = \left[\ \hat{x}_0^T \quad \bar{v}_0^T \quad \bar{n}_0^T\ \right]^T \tag{5}$$

$$P_0^a = \begin{bmatrix} P_{x_0} & 0 & 0 \\ 0 & R_v & 0 \\ 0 & 0 & R_n \end{bmatrix} \tag{6}$$

2) State estimation for $k = 1 \ldots \infty$:

a) Calculate Sigma-points:

$$X_{k-1}^a = \left[\ \hat{x}_{k-1}^a \quad \hat{x}_{k-1}^a + \gamma\sqrt{P_{k-1}^a} \quad \hat{x}_{k-1}^a - \gamma\sqrt{P_{k-1}^a}\ \right] \tag{7}$$

b) Time-update:

$$X_{i,k|k-1}^x = f(X_{i,k-1}^x, X_{k-1}^v, u_{k-1}), \quad i = 0,\ldots,2L$$
$$\hat{x}_k^- = \sum_{i=0}^{2L} w_i^m X_{i,k|k-1}^x \tag{8}$$
$$P_{x_k}^- = \sum_{i=0}^{2L} w_i^c (X_{i,k|k-1}^x - \hat{x}_k^-)(X_{i,k|k-1}^x - \hat{x}_k^-)^T$$

c) Measurement-update:

$$Y_{i,k|k-1} = h(X_{i,k|k-1}^x, X_{k-1}^n), \quad i = 0,\ldots,2L$$
$$\hat{y}_k^- = \sum_{i=0}^{2L} w_i^m Y_{i,k|k-1}$$
$$P_{\hat{y}_k} = \sum_{i=0}^{2L} w_i^c (Y_{i,k|k-1} - \hat{y}_k^-)(Y_{i,k|k-1} - \hat{y}_k^-)^T \tag{9}$$
$$P_{x_k y_k} = \sum_{i=0}^{2L} w_i^c (X_{i,k|k-1}^x - \hat{x}_k^-)(Y_{i,k|k-1} - \hat{y}_k^-)$$

$$K_k = P_{x_k y_k} P_{\hat{y}_k}^{-1}$$
$$\hat{x}_k = \hat{x}_k^- + K_k (y_k - \hat{y}_k^-) \tag{10}$$
$$P_{x_k} = P_{x_k}^- - K_k P_{\hat{y}_k} K_k^T$$

where $x_k^a = \left[\ x_k^T\ v_k^T\ n_k^T\ \right]^T$ and $X_k^a = \left[\ (X_k^x)^T\ (X_k^v)^T\ (X_k^n)^T\ \right]^T$ are the L dimension vectors of the augmented states and Sigma-points respectively, R_v and R_n are the covariances of the process and measurement noise respectively, $\gamma = \sqrt{L+\lambda}$, $\lambda = \alpha^2(L+\kappa)+L$, $\alpha = 10^{-3}$, $\kappa = 0$, $\beta = 2$, and w_i is the weighting parameter defined in *Eq.(2)*.

The Central-Difference KALMAN Filter

This approach is based on STERLING's polynomial interpolation, which was used in the derivations of divided difference filter [10, 11] and central difference filter in Ref. [12]. The result of this approximation can be used for sigma-point approach as in the case of SUT. The algorithm consists of the following steps:

1) Weighted Sigma-point selection:

$$X_0 = \bar{x}$$
$$X_i = \bar{x} + \left(h\sqrt{P_x}\right)_{i=1\ldots L} \tag{11}$$
$$X_i = \bar{x} - \left(h\sqrt{P_x}\right)_{i=L+1\ldots 2L}$$

with weights

$$w_0^m = \frac{h^2-L}{h^2},\ w_i^m = \frac{1}{2h^2},\ w_i^{c_1} = \frac{1}{4h^2},\ w_i^{c_2} = \frac{h^2-1}{4h^2}, \tag{12}$$

where w_i^m, and w_i^c are associated with $i = 1, \ldots, 2L$ sigma-point.

2) Initialization: the same as defined in *Eqs.(4-6)*

3) State estimation for $k = 1, \ldots, \infty$:

a) Sigma-points for time-update:

$$\hat{x}_{k-1}^{a_v} = \left[\ \hat{x}_{k-1} \quad \bar{v}\ \right],\ P_{k-1}^{a_v} = \begin{bmatrix} P_{x_{k-1}} & 0 \\ 0 & R_v \end{bmatrix} \tag{13}$$

$$X_{k-1}^{a_v} = \left[\ \hat{x}_{k-1}^{a_v} \quad \hat{x}_{k-1}^{a_v} + h\sqrt{P_{k-1}^{a_v}} \quad \hat{x}_{k-1}^{a_v} - h\sqrt{P_{k-1}^{a_v}}\ \right] \tag{14}$$

b) Time-update:

$$X_{i,k|k-1}^x = f(X_{i,k-1}^x, X_{k-1}^v, u_{k-1}), \quad i = 0,\ldots,2L$$
$$\hat{x}_k^- = \sum_{i=0}^{2L} w_i^m X_{i,k|k-1}^x$$
$$P_{x_k}^- = \sum_{i=1}^{L} w_i^{c_1} (X_{i,k|k-1}^x - X_{L+i,k|k-1}^x)^2 + \tag{15}$$
$$\sum_{i=1}^{L} w_i^{c_2} (X_{i,k|k-1}^x + X_{L+i,k|k-1}^x - 2X_{0,k|k-1}^x)^2$$

c) Sigma-points for measurement-update:

$$\hat{x}_{k-1}^{a_n} = \begin{bmatrix} \hat{x}_k^- & \bar{n} \end{bmatrix}, P_{k|k-1}^{a_n} = \begin{bmatrix} P_{x_k}^- & 0 \\ 0 & R_n \end{bmatrix} \quad (16)$$

$$X_{k|k-1}^{a_n} = \begin{bmatrix} \hat{x}_{k|k-1}^{a_n} & \hat{x}_{k|k-1}^{a_n} + h\sqrt{P_{k|k-1}^{a_n}} & \hat{x}_{k|k-1}^{a_n} - h\sqrt{P_{k|k-1}^{a_n}} \end{bmatrix} \quad (17)$$

d) Measurement-update:

$$Y_{i,k|k-1} = h(X_{i,k|k-1}^{x}, X_{k|k-1}^{n}), \qquad i = 0,\ldots,2L$$

$$\hat{y}_k^- = \sum_{i=0}^{2L} w_i^m Y_{i,k|k-1}$$

$$P_{\hat{y}_k} = \sum_{i=0}^{2L} w_i^{c_1} (Y_{i,k|k-1} - Y_{L+i,k|k-1})^2 + \sum_{i=0}^{2L} w_i^{c_2} (Y_{i,k|k-1} + Y_{L+i,k|k-1} - 2Y_{0,k|k-1})^2 \quad (18)$$

$$P_{x_k y_k} = \sqrt{w_1^{c_1} P_{x_k}^-} (Y_{1:L,k|k-1} - Y_{L+1:2L,k|k-1})^T$$

$$K_k = P_{x_k y_k} P_{\hat{y}_k}^{-1}$$

$$\hat{x}_k = \hat{x}_k^- + K_k (y_k - \hat{y}_k^-)$$

$$P_{x_k} = P_{x_k}^- - K_k P_{\hat{y}_k} K_k^T$$

Symmetry-Preserving Observers

Among many problems in control theory, the symmetric nature of the system can be utilized, such as in the case of optimal control for feedback or regulations. The symmetry properties can also be used in the design of observers. Symmetry-preserving observers are also known as invariant observers that are based on the differential geometric background of abstract LIE-groups [15, 16, 18].

The constructive algorithm can be defined for designing invariant observers, which is based on the group transformation of symmetry. The aim is to produce invariant frames and invariant output errors to transform a locally asymptotically convergent observer around an equilibrium point into an invariant one, retaining its first-order approximation. The method in Refs. [16, 18] benefits from the fact that the error equations and the first-order approximation can be calculated explicitly and the relationships can be defined globally. In addition, the invariant error equations make stability issues easier to deal with [19-21].

Assumptions

Suppose G is a LIE-group with identity e on manifold Σ. The group transformation $(\varphi_g)_{g \in G}$ on Σ is a smooth mapping

$$(g,\xi) \in G \times \Sigma \rightarrow \varphi_g(\xi) \in \Sigma \quad (19)$$

such that $\varphi_e(\xi) = \xi$ and $\varphi_{g_2}(\varphi_{g_1}(\xi)) = \varphi_{g_2 g_1}(\xi)$ for ζ, g_1, and g_2. The transformation group is local if $\varphi_g(\xi)$ is defined only when g lies sufficiently near to e. We consider only local transformations.

Invariant observers are designed for the smooth non-linear system

$$\tfrac{d}{dt} x = f(x,u), \; y = h(x,u) \quad (20)$$

where x, u, and y belong to the open subsets $X \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$, and $Y \subset \mathbb{R}^p$ respectively and $p \leq n$. Let $r \leq n$ be the dimension of the LIE-group. We assume that for each x the mapping $x \rightarrow \phi_g(x)$ is full rank. Time functions $u(t)$ and $y(t)$ are known.

Invariant Pre-Observer

For the differential geometric description of the symmetric observers [14], the following definitions and theorems should be introduced:

Definition 1. The group of local transformation on manifold $X \times U$ is expressed by

$$(X,U) = (\phi_g(x), \psi_g(u)) \quad (21)$$

where $\phi_g(x), \psi_g(u)$ is local diffeomorphism and $g \in G$

Definition 2. System $\tfrac{d}{dt} x = f(x,u)$ is G-invariant, if $f(\phi_g(x), \psi_g(u)) = D\phi_g(x) \cdot f(x,u)$ for g, x, u. Thus, the system remains unchanged for the local transformation, $\dot{X} = f(X,U)$.

Definition 3. The output $y = h(x,u)$ is G-equivariant, if there exists ρ_g transformation on Y, such that $g \in G$ and $h(\phi_g(x), \psi_g(u)) = \rho_g(h(x,u))$ for g, x, u. Thus, the output remains unchanged for the local transformation, $Y = h(X,U)$.

Definition 4. The vector field w is G-invariant on manifold X, if $\tfrac{d}{dt} x = w(x)$ is invariant to the transformation, i.e. $w(\phi_g(x)) = D\phi_g(x) \cdot w(x)$ for g, x.

Definition 5. An invariant frame (w_1, w_2, …, w_3) consists of n linearly point-wise independent G-invariant vector fields on X thus $(w_1(x), w_2(x), \ldots, w_n(x))$ is a basis of the tangent space $T_X(x)$.

Lemma 1. The invariant frame can be constructed from the canonical basis of X. The vector field defined by

$$w_i(x) = (D\phi_{\gamma(x)})^{-1} \cdot \tfrac{\partial}{\partial x_i} \quad (22)$$

for i = 1, …, n forms an invariant frame.

Definition 6. The system $\tfrac{d}{dt} \hat{x} = F(\hat{x}, u, y)$ is a pre-observer, if $F(x,u,h(x,u)) = f(x,u)$ is satisfied for x and u. Furthermore, if $\lim \hat{x}(t) = x(t)$ as $t \rightarrow \infty$, then the pre-observer is asymptotic.

Definition 7. The pre-observer is G-invariant, if

$$F(\varphi_g(\hat{x}),\psi_g(u),\rho_g(y)) = D\varphi_g(\hat{x}) \cdot F(\hat{x},u,y) \tag{23}$$

is satisfied for g, $\hat{x}$, u, and y i.e. the observer remains unchanged for the group transformation on $\mathrm{X},\mathrm{U},\mathrm{Y}$.

Thus, $X = \phi_g(x)$, $U = \psi_g(u)$, $Y = \rho_g(y)$, and $\frac{d}{dt}\hat{X} = F(\hat{X},U,Y)$.

An invariant observer is an asymptotic G-invariant pre-observer. Moreover, if the pre-observer is G-invariant and $\operatorname{rank}(F) = \dim y$, then the output map y is G-equivariant.

Definition 8. Smooth map $(\hat{x},u,y) \to E(\hat{x},u,y) \in \mathbb{R}^p$ is an invariant output error if

a) $y \to E(\hat{x},u,y)$ is invertible for $\hat{x}$, u, and y

b) $E(\hat{x},u,h(\hat{x},u)) = 0$ for $\hat{x}$ and u

c) $E(\phi_g(\hat{x}),\psi_g(u),\rho_g(y)) = E(\hat{x},u,y)$ for $\hat{x}$, u, and y.

The first two properties mean E is an output error, the third expresses the invariance. The introduction of the invariant error is necessary, because the output error $\hat{y} - y$ does not usually preserve the geometry of the system [21].

The following three theorems [15, 16] support the construction of the symmetry-preserving observers.

Theorem 1. The observer $\frac{d}{dt}\hat{x} = F(\hat{x},u,y)$ is a G-invariant pre-observer for the G-invariant non-linear *Eq.(20)* with a G-equivariant output if and only if

$$F(\hat{x},u,y) = f(\hat{x},u) + \sum_{i=1}^{n} L_i(I(\hat{x},u),E(x,u,y))w_i(\hat{x}) \tag{24}$$

where E is an invariant output error, $(\hat{x},u) \to I(\hat{x},u) \in \mathbb{R}^{n+m-r}$ is a full rank function, L_i is a smooth function, such that $L_i(I(\hat{x},u),0) = 0$ and $(w_1,w_2,\ldots,w_n)$ is an invariant frame. Since $L_i(I,E) = \bar{L}_i(I,E)E$

$$\sum_{i=1}^{n} L_i(I,E)w_i = \sum_{i=1}^{n} w_i(\bar{L}_i(I,E)E)$$

$$= \begin{pmatrix} w_1 & \ldots & w_n \end{pmatrix} \begin{pmatrix} \bar{L}_1(I,E) \\ \vdots \\ \bar{L}_n(I,E) \end{pmatrix} E \tag{25}$$

The observer can be written as follows

$$F(\hat{x},u,y) = f(\hat{x},u) + W(\hat{x})L(I(\hat{x},u),E(\hat{x},u,y))E(\hat{x},u,y). \tag{26}$$

Theorem 2. The following statements are valid:

- $(\hat{x},u,y) \to E(\hat{x},u,y)$ invariant output error exists,
- there is a $(\hat{x},u) \to I(\hat{x},u) \in \mathbb{R}^{n+m-r}$ invariant function,
- every other invariant output error has the form of

$$\tilde{E}(\hat{x},u,y) = L(I(\hat{x},u),E(\hat{x},u,y)) \tag{27}$$

where L is a smooth function such that $L(I, 0) = 0$ and $E \to L(I,E)$ is invertible.

Theorem 3. Consider the equilibrium of a non-linear system

$$f(\bar{x},\bar{u}) = 0 \text{ and } \bar{y} = h(\bar{x},\bar{u})$$

Assume that the linearized system A,B,C,D around this equilibrium is observable, where

$$A = \frac{\partial f}{\partial x}(\bar{x},\bar{u}),\ B = \frac{\partial f}{\partial u}(\bar{x},\bar{u}),\ C = \frac{\partial h}{\partial x}(\bar{x},\bar{u}),\ A = \frac{\partial h}{\partial u}(\bar{x},\bar{u})$$

and let L be such that $A + LC$ is a stable matrix. From the locally asymptotically convergent observer

$$\frac{d}{dt}\hat{x} = f(\hat{x},u) + L(\hat{y} - y),\ \hat{y} = h(\hat{x},u) \tag{28}$$

an invariant observer can be constructed by the same linear approximation

$$\frac{d}{dt}\hat{x} = f(\hat{x},u) + W(\hat{x})\bar{L}(I(\hat{x},u),E(\hat{x},u,y))E(\hat{x},u,y) \tag{29}$$

where $\bar{L} = -W(x)^{-1}LV^{-1}$ and $V = \frac{\partial E}{\partial y(\bar{x},\bar{u},\bar{y})}$ is an invertible matrix of $p \times p$ dimensions.

There is no general algorithm for designing the L_i gain functions of *Theorem 1*. However, if we consider the following invariant state error

$$\eta(x,\hat{x}) = \phi_{\gamma(x)}(\hat{x}) - \phi_{\gamma(x)}(x) \tag{30}$$

instead of the $\hat{x} - x$ state error, where $\gamma(x)$ is the solution of

$$\phi_g^a(x) = c \tag{31}$$

with respect to g, then the invariant error will only depend on the $I(x,u)$ trajectories.

Constructive Algorithm

Consider an invariant system, i.e. unchanged by transformation *Eq.(21)* with an equivariant output (*Definition 3*). The non-linear observer design for non-linear dynamic systems [10, 11] can be divided into the following steps:

a) Choose a G LIE-group and a group transformation taking into account the symmetrical properties of the system.
b) Solve the normalization *Eq.(31)*. Build an invariant error E and the complete set of scalar invariants I.
c) Construct the invariant frame *Eq.(22)*.
d) Determine the pre-observer *Eq.(25)*.
e) Linearize the system $f(\bar{x},\bar{u}) = 0$ around the equilibrium and obtain A,B,C,D matrices. Check the observability and design an invariant observer from the chosen linear observer by using *Theorem* 3. Choose an appropriate $\bar{L}$.
f) Choose the parameters of the linearized error equations, based on the invariant state-errors η, such that the invariant system will be asymptotically stable.

Then the non-linear symmetry-preserving observer is asymptotically stable, and converges locally and exponentially along all system trajectories.

Case Study: A Non-Holonomic Vehicle

In this section, the previously introduced state estimation procedures are employed for an example of a non-holonomic vehicle tracking control. The well-known model in the robotics of the simplified vehicle kinematics is considered. The tracking is maintained by satisfying the kinematic constraints with the appropriate choice of reference signals.

Kinematic Model

The simplified kinematic model of the vehicle can be written in the following form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \\ uv \end{pmatrix}, \; h(x,y,\theta) = (x,y) \tag{32}$$

where u is the velocity, v is the curvature, i.e. a function of the steering angle and h is the measured output. All signals contain additive Gaussian noise. The model assumes that the contact point of the wheel and the ground does not slip. Notice, that the linearized system is not observable.

Tracking

For path tracking the control signals are set to satisfy the kinematic constraints, thus u and v are calculated as follows:

$$u(t) = \sqrt{\dot{x}_r^2(t) + \dot{y}_r^2(t)} \text{ and } v(t) = \frac{\ddot{y}_r(t)\dot{x}_r(t) - \dot{x}_r(t)\ddot{y}_r(t)}{u^3(t)} \tag{33}$$

where $x_r(t)$ and $y_r(t)$ are time functions of the reference signal. The initial conditions for tracking control is

$$\theta_r(0) = \tan^{-1}\left(\frac{\dot{y}_r(0)}{\dot{x}_r(0)}\right). \tag{34}$$

Hence, the reference signals for the vehicle velocity and steering function can be calculated. In the simulations, sinusoidal path tracking is implemented using the reference signals

$$x_r(t) = t, \text{ and } y_r(t) = A\sin(\omega t).$$

Invariant Observer Design

Group Transformation

Eq.(32) is independent of the origin and of the orientation of the frame chosen, i.e. it is invariant with regards to the group of rotations and translations. We chose the Lie-group $G = SE(2)$ and defined the group transformation as follows:

$$\begin{aligned} \phi_{(x_g,y_g,\theta_g)}(x,y,\theta) &= \begin{pmatrix} x_g \\ y_g \\ \theta_g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \\ &= \begin{pmatrix} x\cos\theta_g - y\sin\theta_g + x_g \\ x\sin\theta_g + y\cos\theta_g + y_g \\ \theta + \theta_g \end{pmatrix} \end{aligned} \tag{35}$$

$$\psi_{(x_g,y_g,\theta_g)}(u,v) = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(x_g,y_g,\theta_g)}(x,y) = \begin{pmatrix} x\cos\theta_g - y\sin\theta_g + x_g \\ x\sin\theta_g + y\cos\theta_g + y_g \end{pmatrix}$$

One can check that *Eq.(32)* is invariant to the transformations φ, and ψ in terms of *Definition 1* and the output is equivariant to transformation ρ in terms of *Definition 3*.

Invariant Output Errors

The normalization *Eq.(31)* for $c = 0$:

$$\begin{aligned} 0 &= x\cos\theta_\gamma - y\sin\theta_\gamma + x_\gamma \\ 0 &= x\sin\theta_\gamma + y\cos\theta_\gamma + y_\gamma \\ 0 &= \theta + \theta_\gamma \end{aligned} \tag{36}$$

The solution of *Eq.(36)*, the scalar invariants and output errors can be written as follows respectively

$$\begin{pmatrix} x_\gamma \\ y_\gamma \\ \theta_\gamma \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}^{-1} = \begin{pmatrix} -x\cos\theta - y\sin\theta \\ x\sin\theta - y\cos\theta \\ -\theta \end{pmatrix} = \gamma \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \tag{37}$$

$$I(x,y,\theta,u,v) = \psi_{\gamma(x,y,\theta)}(u,v) = \begin{pmatrix} u \\ v \end{pmatrix} \tag{38}$$

$$\begin{aligned} E &= \rho_{(x_\gamma,y_\gamma,\theta_\gamma)}(\hat{x},\hat{y}) - \rho_{(x_\gamma,y_\gamma,\theta_\gamma)}(x,y) \\ &= \begin{pmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix} \begin{pmatrix} \hat{x} - x \\ \hat{y} - y \end{pmatrix} \end{aligned} \tag{39}$$

Invariant Frame

$$\begin{aligned} W &= (D\phi_{\gamma(x,y,\theta)}(x,y,\theta))^{-1} \tfrac{\partial}{\partial(x,y,\theta)} = \\ &= D\phi_{\gamma(x,y,\theta)}(x,y,\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{40}$$

Invariant pre-observer

$$\frac{d}{dt}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{\theta}\end{pmatrix}=\begin{pmatrix}u\cos\hat{\theta}\\ u\sin\hat{\theta}\\ uv\end{pmatrix}+\begin{pmatrix}\cos\hat{\theta} & -\sin\hat{\theta} & 0\\ \sin\hat{\theta} & \cos\hat{\theta} & 0\\ 0 & 0 & 1\end{pmatrix}\overline{L}\times \begin{pmatrix}\cos\hat{\theta} & \sin\hat{\theta}\\ -\sin\hat{\theta} & \cos\hat{\theta}\end{pmatrix}\begin{pmatrix}\hat{x}-x\\ \hat{y}-y\end{pmatrix} \quad (41)$$

where $\overline{L}$ is a smooth 3×2 gain matrix, whose elements depend on the invariant error *E* and the invariants *I.*

Error Equations and Error Dynamics

$$\eta=\begin{pmatrix}\eta_x\\ \eta_y\\ \eta_\theta\end{pmatrix}=\gamma(\hat{x},\hat{y},\hat{\theta})\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{\theta}\end{pmatrix}-\gamma(\hat{x},\hat{y},\hat{\theta})\begin{pmatrix}x\\ y\\ \theta\end{pmatrix}= =\begin{pmatrix}(\hat{x}-x)\cos\hat{\theta}+(\hat{y}-y)\sin\hat{\theta}\\ -(\hat{x}-x)\sin\hat{\theta}+(\hat{y}-y)\cos\hat{\theta}\\ (\hat{\theta}-\theta)\end{pmatrix} \quad (42)$$

$$\begin{pmatrix}\dot{\eta}_x\\ \dot{\eta}_y\\ \dot{\eta}_\theta\end{pmatrix}=\begin{pmatrix}u(1-\cos\eta_\theta)+(uv+\overline{L}_{31}\eta_x+\overline{L}_{32}\eta_y)\eta_y\\ u\sin\eta_\theta-(uv+\overline{L}_{31}\eta_x+\overline{L}_{32}\eta_y)\eta_x\\ 0\end{pmatrix}+ +\overline{L}\begin{pmatrix}\eta_x\\ \eta_y\end{pmatrix} \quad (43)$$

It can be seen that the invariant error equations are independent of the trajectory and only depend on the relative quantities η_x, η_y, and η_θ and the *I* invariants.

Stability and Convergence

The weighting matrix is defined as follows

$$\overline{L}=\begin{pmatrix}-|u|a & ubE_y-uv\\ uv-ubE_y & -|u|c\\ 0 & -ub\end{pmatrix}, \quad (44)$$

where *a* and *b* are *c*are positive scalars and $E_y = \eta_y$. The invariant error dynamics can be made locally asymptotically stable. Furthermore, the resulting symmetry-preserving observer is almost globally asymptotically convergent, i.e. it converges any initial condition except for one as described in Refs. [15-16].

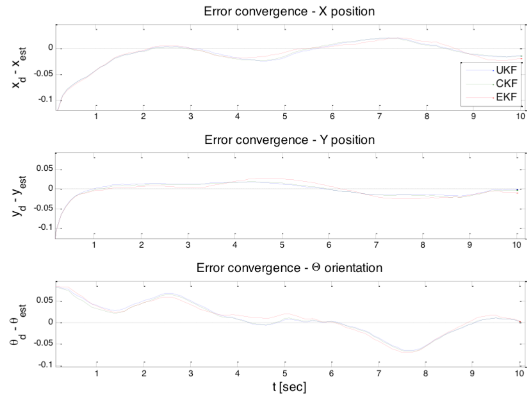

Figure 2: Estimation errors of the sigma-point and extended KALMAN filters from Example 1

Simulation Results

The state estimation techniques are illustrated in two tracking examples. For comparative purposes, the following algorithms have been implemented: Extended KALMAN Filter (EKF), Central Difference and Unscented KALMAN Filter (CKF, UKF) and the symmetry preserving observer (SYM) discretized by the sample time of the sigma-point estimators.

The reference paths are $x_r(t)=t$, and $y_r(t)=A\sin(\omega t)$, where A = 3 m and ω = π/5 rad s^{-1}. Zero mean additive Gaussian noise is assumed with standard deviations of 0.01 and 0.005 for the process noise and 0.02 for the measurement noise. The sampling time is T_s = 0.1 s. The initial values are $x_0^{est}=y_0^{est}=\theta_0^{est}=1$, and $x_0=y_0=0$.

Example 1

The vehicle starts from the origin of the *xy*-plane with large initial estimation error and tracks a sinusoidal path. *Fig.1* shows the estimated output positions of the vehicle using sigma-point KALMAN filters and symmetry-preserving observers. The estimation errors of different algorithms can be seen in *Figs.2* and *3*.

The errors are smaller and show smoother signals in the case of the SPKF estimators, which was expected, since the SYM observer is deterministic. Large errors appear at the peak values of the reference signals; however, both estimation techniques achieved the same satisfactory results in spite of the relatively small sampling frequency.

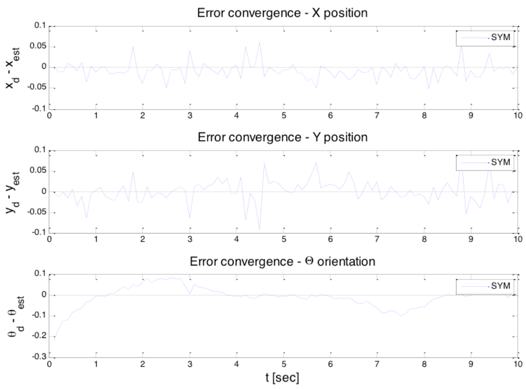

Figure 3: Estimation errors of the symmetry-preserving observer (SYM) from Example 1

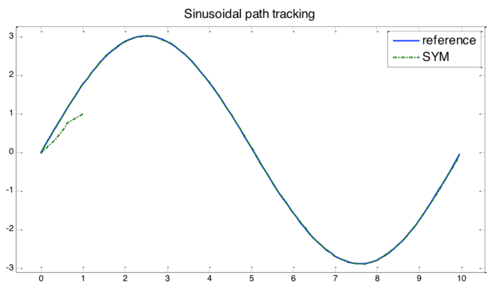

Figure 5: Estimated output position of the vehicle using symmetry-preserving observer (SYM) from Example 2

Table 1: Root-mean-squared errors of the estimated signals from Example 1

RMSE	*x*, m	*y*, m	θ, rad
UKF	0.1368	0.1587	0.0338
CKF	0.1366	0.1577	0.0336
EKF	0.1371	0.1551	0.0352
SYM	0.1371	0.1628	0.0455

Table 2: Root-mean-square errors of the estimated signals from Example 2

RMSE	*x*, m	*y*, m	θ, rad
UKF	0.0998	0.1180	0.0627
CKF	0.0993	0.1175	0.0628
EKF	0.1013	0.1165	0.0648
SYM	0.1264	0.1453	0.0760

Root-mean-square errors can be seen in *Table 1*. We can conclude that among the processes of state estimation techniques the SYM observer is competitive with SPKF estimators.

Example 2

In this example the vehicle performs an off-line estimation before movement starts, followed by an on-line estimation along a sinusoidal path. It is notable that in the stationary state the velocity is zero and at the beginning of movement the velocity changes to a non-zero value abruptly, see *Eq.(33)*. *Figs.4-7* illustrate the estimated output positions and the measurement errors for the sigma-point estimators and the invariant observers respectively

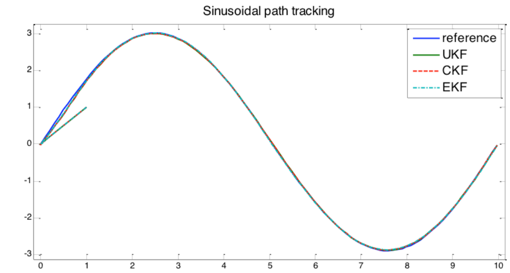

Figure 4: Estimated output position of the vehicle using sigma-point state estimation (UKF, CKF) and extended KALMAN Filter (EKF) from Example 2

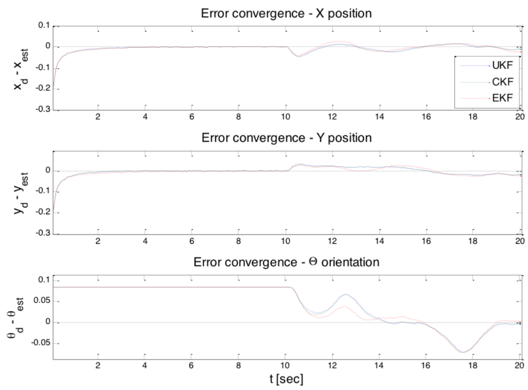

Figure 6: Estimation errors of the sigma-point and the extended KALMAN filters from Example 2

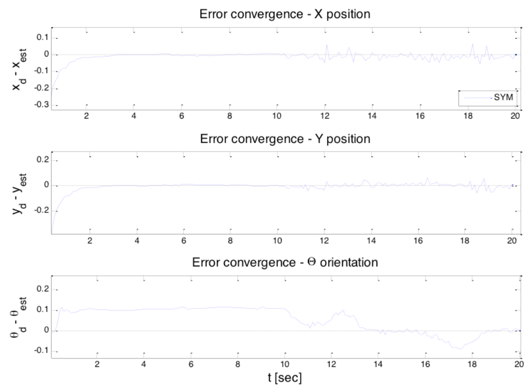

Figure 7: Estimation errors of the symmetry-preserving observer (SYM) from Example 2

The error signals show a quick convergence in the case of the symmetry-preserving observer. The speed of convergence depends on the parameters set in the $\bar{L}$ matrix; however, high gains can cause the observer to be divergent and make larger errors. Since there are no u and δ signals in the stationary position and the orientation is not directly measured, the orientation can only be estimated after the vehicle starts moving.

The error signal oscillates around zero in a similar way to Example 1. Due to calibration, smaller errors appear. The invariant observer and sigma-point estimators have the same magnitude of error components. The root-mean-square errors of the simulation can be seen in *Table 2*.

Conclusion

In this paper we presented a comparison of two different recently developed state estimation techniques. The Sigma-Point (SPKF) stochastic estimator is based on the Unscented Transformation and the non-linear symmetry-preserving observers are based on the group transformation of symmetrical LIE groups and invariant frames. Two examples are considered for the estimation of vehicle kinematics: the first example deals with the path tracking reliability of the vehicle; the second example examines the convergence and the speed of the off-line calibration followed by path tracking.

The examples illustrate that for non-linear systems in noisy environments the use of stochastic state-estimators is needed. Moreover the Sigma-point approach provides more accurate estimation results than the commonly used non-linear EKF state estimators.

The symmetry preserving observers for non-linear systems benefit from their constructive design method, which provides local asymptotic convergence. This method can be utilized by non-linear applications in which the physical and mechanical symmetry properties of the system can be exploited.

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